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An Efficient Algorithm for Solving the Inverse Problem of Locating the Interfaces Using the Frequency Sounding Data

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We consider the inverse coefficient problem of locating the interface positions arising in frequency sounding of layered media. Such a problem is of particular interest in the exploration of geophysics, underwater acoustics and electromagnetics, optical sensing, and so forth. We found that a simplified algorithm can be constructed to determine the approximate positions of interfaces. Unlike the conventional nonlinear least squares, this algorithm does not require the time-consuming constrained optimisation. Instead, the predictor–corrector method is applied to solve numerically the auxiliary Cauchy problem for a Riccati equation. The feasibility of this algorithm is demonstrated in computational experiments with a model problem of electromagnetic frequency sounding of layered media.

Key Words: MT frequency sounding; inverse coefficient problem; locating the interfaces; Riccati equation.

1. INTRODUCTION

Apparently, the term *sounding* was first introduced in the exploration of geophysics in connection with using the natural or controlled sources of the electromagnetic field for probing Earth's crust (see, e.g., [1, 2]). One of the most used modes of sounding, frequency sounding, consists of determining the spatial distribution of material properties, e.g., the conductivity, from boundary observations of the frequency dependence of certain functionals of the reflected field. Generally, the spatial distribution of a suitable material property recovered from frequency sounding data includes information about the geometry of inhomogeneities. However, in the case of a layered model, it is meaningful to consider two sequential problems. The first problem consists of locating the interfaces between layers, whereas the second problem consists of determining the spatial distribution of material properties inside each layer. It is clear that the solution of the first problem is treated

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as *a priori* information when solving the second problem. The goal of this paper is to present a simple, but efficient, numerical technique for estimating the interface locations from frequency sounding data. Although we focus mainly on electromagnetic frequency sounding, the proposed technique can also be applied with minor modifications to the analogous problems arising in acoustic and seismic inversions and optical sensing.

It is recognised by the geophysics community that even for relatively simple layered configurations, the conventional least-squares solution depends strongly on *a priori* information about the stratification of such configurations (e.g., a number of layers) and on a method of optimisation (see, e.g., [3]). For instance, it is well known (see, e.g., [4]) that increasing the number of layers in a model configuration may generate several artefacts. In particular, the spurious thin conductive layers may appear in the recovered conductivity profile. This fact can be explained by the multiextremality of the residual objective functions, i.e., they may have many local minima. In many realistic cases, the multiextremality of such objective functions is more typical than exceptional (see, e.g., [3]), especially when dealing with incomplete and noisy data. Under these conditions, the numerical techniques (see, e.g., [5, 6]) based on the gradient or Newton-like methods may fail if the starting vector is improperly chosen. It should also be pointed out that applying the so-called damped (regularised) least squares (see, e.g., [7]), i.e., actually Lagrange's or Tikhonov's smoothing functionals, cannot guarantee the strict convexity of the corresponding objective function.

In the mathematics literature, there are two general approaches to global optimisation. The first approach is to construct the global search algorithms, such as the Lipschitzian optimisation algorithms (see, e.g., [8]), genetic algorithms [9], and simulated annealing algorithms [10]. The second approach is to convexify the originally multiextremal objective function, allowing the use of numerical methods of convex programming. This approach has been recently developed using either Carleman's weight functions [11–14] or the layer stripping technique [16]. Obviously, in the case of a layered medium, any of these approaches could also be used for locating the interfaces. However, splitting the general inverse problem of frequency sounding into the two sequential problems indicated above allows us to consider separately the problem of locating the interfaces without knowing the material properties. It becomes possible due to our finding that the points of discontinuity of the conductivity profile $\sigma(z)$ imply the critical points of the second derivative of a certain function. This function satisfies an auxiliary boundary value problem for a Riccati equation that resulted from some identical transformations of the original boundary value problem (see [11–15]), and the physically meaningful conjectures on the solution of the auxiliary problem. As a result, the Riccati equation does not contain the unknown function $\sigma(z)$, and the Cauchy condition is determined from the frequency sounding data. Thus, the problem of locating the interfaces can be reduced to computing the numerical solution of the Cauchy problem for the Riccati equation and determining the corner points of its solution. After the interfaces are located, one can solve the problem of determining the unknown conductivity vector via any technique indicated above or even via any regularised gradient of Newton-like methods. However, in this paper we do not address this problem. We focus mainly on the problem of locating the interfaces.

It should also be pointed out that in Section 1.3 we make some conjectures when reducing the overdetermined boundary value problem for an integrodifferential equation to the Cauchy problem for a Riccati equation. Although these conjectures are not rigorously proven, they are physically meaningful for at least low-frequency electromagnetic sounding of layered conductive configurations, and the numerical study demonstrates their feasibility.

The paper is arranged as follows. In Section 2, we formulate the problems and justify the basic propositions. In Section 3, we construct a simplified algorithm for locating the interfaces. In Section 4, to demonstrate the feasibility of the proposed approach, we perform some computational experiments with the model problem of frequency electromagnetic sounding of layered media. In a short Section 5, we conclude the paper.

2. THEORY

2.1. *Problem Formulation*

For brevity, we restrict our consideration to the 1-D model of magnetotelluric (MT) frequency sounding of a layered conductive medium at sufficiently low frequencies. Specifically, consider the three-layer configuration whose electrical conductivity distribution is

$$\sigma(z) = \begin{cases} 0, & \text{for } z < 0, \\ \sigma(z), & \text{for } 0 < z < L, \\ \sigma_b, & \text{for } z > L, \end{cases} \tag{1}$$

where $\sigma(z) \geq const > 0$ is an arbitrary piecewise continuous function and $\sigma_b = const > 0$, such that $\sigma_b \ll \sigma(z)$. In MT sounding, the electromagnetic field of distant natural sources varies very slowly in the horizontal directions on Earth's surface. It is, therefore, assumed that a source is a plane elctromagnetic wave normally incident on the surface $z = 0$, and the electromagnetic field in layered media depends only on depth z. Without loss of generality, we consider the TE mode of MT sounding; i.e., we assume that $\mathbf{E} = (E_x, 0, 0)$, $\mathbf{H} = (0, H_y, H_z)$. Also, we assume that the time factor is $\exp(-\imath\omega t)$. The 1-D model of MT frequency sounding was first formulated and studied by Tikhonov [1] and Cagniard [2] in the early 1950s of the 20th century. Since then, it has been extensively tested against simulated and field data and its consistency with the reality has been well established (see, e.g., [17]).

It is well known from the contemporary geophysics literature (see, e.g., [17]) that the governing equation for the source-free TE field inside an inhomogeneous layer $0 < z < L$ can be obtained from Maxwell's equations by excluding the magnetic component of the electromagnetic field. The final form is

$$\nabla^2 E_x + (\varepsilon_0 \varepsilon(z)\omega^2 \mu + \imath\omega\mu\sigma(z))E_x = 0, \tag{2}$$

where $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m is the absolute permittivity of vacuum, $\mu = 4\pi \times 10^{-7}$ H/m is the magnetic permeability, which is assumed to be constant everywhere, and $\varepsilon(z)$ is the spatial distribution of relative permittivity in the inhomogeoeus layer. In conductive media at low frequencies, the loss tangent $\tan\delta = \sigma(z)/\varepsilon_0\varepsilon(z)\omega \gg 1$; i.e., the induction is negligibly small. From a physical standpoint, this means that in such media the wave process is not practically realised. Therefore, the term $\varepsilon_0\varepsilon(z)\omega^2\mu$ in (2) is usually neglected (see, e.g., [1, 2]) when deriving the governing equation in the form

$$\nabla^2 E_x + \imath\omega\mu\sigma(z)E_x = 0, \quad 0 < z < L. \tag{3}$$

Assume that the plane incident wave is $E_x^{inc}(z) = E_0 \exp(\imath k_a z)$, where $k_a = \omega\sqrt{\varepsilon_0\mu}$ is the wavenumber in the air. Also, we assume that the electric field $(E_x(z) - E_x^{inc}(z))$ satisfies the radiation condition as $z \to \infty$. Under such assumptions, we derive the boundary conditions for Eq. (3) by exploiting the continuity of tangential components of both the electric and the magnetic fields at $z = 0$ and $z = L$. Since $H_y = (-\imath\omega\mu)^{-1}\partial E_x/\partial z$, we obtain the following boundary conditions:

$$E_x|_{z=0,z>0} = E_x|_{z=0,z<0}, \tag{4}$$

$$\left.\frac{\partial E_x}{\partial z}\right|_{z=0,z<0} = \left.\frac{\partial E_x}{\partial z}\right|_{z=0,z>0}, \tag{5}$$

$$E_x|_{z=L,z<L} = E_x|_{z=L,z>L}, \tag{6}$$

$$\left.\frac{\partial E_x}{\partial z}\right|_{z=L,z<L} = \left.\frac{\partial E_x}{\partial z}\right|_{z=L,z>L}, \tag{7}$$

Introducing the normalised electric field $u(z,\omega) = E_x(z,\omega)/E_x(0,\omega)$ and using the boundary conditions (4), (6), and (7), we derive the boundary value problem governing the TE field inside the inhomogeneous layer. Indeed, it follows from (4) that $u(0,\omega) = 1$. Taking into account the fact that in the homogeneous half-space $z > L$ there exists only the transmission wave $T(\omega)\exp(\imath k_b z)$, where $k_b = \sqrt{\imath\omega\mu\sigma_b}$, both conditions (6) and (7) result in the Robin condition $u'(L,\omega) - \imath k_b u(L,\omega) = 0$. Thus, we arrive at the scalar boundary value problem in the inhomogeneous layer:

$$u'' + \imath\mu\omega\sigma(z)u = 0, \quad 0 < z < L, \tag{8}$$

$$u'(0,\omega) = 1, \tag{9}$$

$$u'(L,\omega) - \imath k_b u(L,\omega) = 0. \tag{10}$$

In MT sounding, the admittance

$$Y(\omega) = \left.\frac{H_y(z,\omega)}{E_x(z,\omega)}\right|_{z=0,z>0} = -(\imath\omega\mu)^{-1}u'(0,\omega)$$

is usually observed on the surface $z = 0$. This fact allows us to separate problem (8)–(10) from the boundary value problem in the homogeneuos half-space $z < 0$ with the Robin boundary condition $u'(0,\omega) - \imath\omega\mu Y(\omega) = 0$.

Let $u \in H^2(0,L)$, where $H^2(0,L)$ is a Sobolev space. Note that problem (8)–(10) is the Sturm–Liouville problem. Therefore, under conditions formulated above, it is uniquely solvable for any $\omega \in [\omega_{min}, \infty)$, where $\omega_{min} > 0$ is a certain real number (see [15] for details).

Let $Z = \{z_\imath\}_{\imath=1}^N$ be a set of points of discontinuity of the conductivity profile $\sigma(z)$. Then, we formulate the problem of locating the interfaces as follows.

Given the function $u'(0,\omega) = -\imath\omega\mu Y(\omega)$, *and the normalised electric field*, $u(z,\omega)$ *satisfies the boundary value problem* (8)–(10). *Determine the set Z.*

2.2. *Nonlinear Least Squares*

Consider the inverse problem posed above. Since the nonlinear operator generated by this problem depends on the unknown distribution of conductivity $\sigma(z)$, applying the conventional nonlinear least squares leads to a minimising of the residual objective function on a set of admissible functions $\sigma(z)$. Moreover, our main goal is to determine the set Z rather than the function $\sigma(z)$. Therefore, it is natural to reformulate the original inverse problem in terms that allow eliminating the unknown function $\sigma(z)$ from the differential equation, preserving only the implicit dependence of $u(0;\omega)$ on $\sigma(z)$. Such a procedure was recently developed [14, 15] as a part of the convexification approach.

We first transform the original inverse problem, introducing the new function $v(z;\omega) = \ln u(z;\omega)$. We obtain

$$v'' + (v')^2 = -\imath\omega\mu\sigma(z), \quad 0 < z < L, \tag{11}$$

plus three boundary conditions at $z = 0$ and $z = L$. Introducing another function $q(z;\omega) = \frac{\partial}{\partial\omega}(\frac{v}{\omega})$ (and, hence, $v(z;\omega) = -\omega\int_\omega^\infty q(z;\psi)\,d\psi$), we then obtain the overdetermined boundary value problem

$$q'' - 2\omega q'\int_\omega^\infty q'\,d\psi + \left(\int_\omega^\infty q'\,d\psi\right)^2 = 0, \quad 0 < z < L, \tag{12}$$

$$q(0;\omega) = \varphi_1(\omega), \tag{13}$$

$$q'(0;\omega) = \varphi_2(\omega), \tag{14}$$

$$q'(L;\omega) = \varphi_3(\omega), \tag{15}$$

where

$$\varphi_1(\omega) = \frac{\partial}{\partial\omega}\left[\frac{\ln u(0;\omega)}{\omega}\right], \tag{16}$$

$$\varphi_2(\omega) = -\frac{\partial}{\partial\omega}\left[\frac{2}{\omega u(0;\omega)}\right], \tag{17}$$

$$\varphi_3(\omega) = \frac{\sqrt{2\mu\sigma_b}}{2}\frac{1-\imath}{\omega^{3/2}}. \tag{18}$$

It can be seen that the integrodifferential operator in (12) does not depend on the unknown function $\sigma(z)$, whereas the boundary conditions at $z = 0$ contain the given function $u(0;\omega)$. Therefore, the inverse model (12)–(15) is more appropriate for further analysis. The existence of integrals in (12) was proven in [14].

To reformulate the problem of locating the interfaces in terms of problem (12)–(15), it is necessary to establish the connection between the least-squares solution of this problem and the sought set Z. Before we embark on an analysis, we first notice that the set Z is the same for both the function $\sigma(z)$ and $u''(z;\omega)$ for all ω. This follows immediately from the integral representation of $u(z;\omega)$ via Green's function for problem (8)–(10). This fact implies the following conjecture. Since the transformation

$$\frac{\partial}{\partial\omega}\left[\frac{\ln u(z;\omega)}{\omega}\right], \quad \omega \neq 0,$$

is continuous, a certain derivative of the least-squares solution $q(z;\omega)$ should also have points of discontinuity coinciding with points $\{z_\imath\}_{\imath=1}^N$. Now, we shall prove this conjecture, showing that precisely the third derivative $q'''(z;\omega)$ has such points.

Let $z_\iota \in Z$ be an arbitrary point. Denote $\sigma_+ = \lim_{z \to z_\iota +} \sigma(z)$ and $\sigma_- = \lim_{z \to z_\iota -} \sigma(z)$. Since $v' = u'/u,\ v'' = u''/u - (u'/u)^2$, then the functions v, v' are continuous at z_ι. However, the function v'' has a discontinuity at z_ι since the function u'' is discontinuous. Then, it follows from (11) that

$$\frac{v''_+ - v''_-}{\omega} = -\iota\mu(\sigma_+ - \sigma_-).$$

Hence,

$$q''_+ - q''_- = \frac{\partial}{\partial\omega}\left[\frac{v''_+ - v''_-}{\omega}\right] = 0.$$

This means that the second derivative q'' is continuous at z_ι. Calculating the thrid derivatives of functions u, v, we obtain

$$u''' = -\iota\omega\mu\sigma' u - \iota\omega\mu\sigma u', \tag{19}$$

$$v''' = u'''/u - 3(u'' \cdot u')/u^2 + 2(u'/u)^3. \tag{20}$$

Since $\sigma'(z) = 0$ almost everywhere, the first term in (19) vanishes at all continuous points, so that we have from (20) at z_ι

$$\frac{v'''_+ - v'''_-}{\omega} = 2\iota\mu(\sigma_+ - \sigma_-)\frac{u'}{u}.$$

This equality implies

$$q'''_+ - q'''_- = 2\iota\mu(\sigma_+ - \sigma_-)\frac{\partial}{\partial\omega}\frac{u'}{u}.$$

This means that the functions $q'''(z;\omega)$, $\sigma(z)$ have the discontinuity at the same point z_ι. This is true for any point $z_\iota \in Z$. In other words, the function q''' and σ have the same set of points of discontinuity.

However, it is impractical to employ the boundary value problem (12)–(15) as an inverse model of frequency sounding, because it requires the frequency sounding data measured at all possible positive frequencies. Meanwhile, the practitioners are usually dealing with sufficiently limited frequency range $[\underline{\omega}, \bar{\omega}]$. In particular, in this paper we consider the low-frequency case. Therefore, we rewrite this problem (12)–(15) in the form

$$q'' - 2\omega q' \int_{\omega}^{\bar{\omega}} q'\, d\psi + \left(\int_{\omega}^{\bar{\omega}} q'\, d\psi\right)^2 - F(q', \omega, \bar{\omega}) = 0, \quad 0 < z < L, \tag{21}$$

$$q(0;\omega) = \varphi_1(\omega), \tag{22}$$

$$q'(0;\omega) = \varphi_2(\omega), \tag{23}$$

$$q'(L;\omega) = \varphi_3(\omega), \tag{24}$$

where

$$F(q', \omega, \bar{\omega}) = 2\omega q' \int_{\bar{\omega}}^{\infty} q'\, d\psi - \left(\int_{\bar{\omega}}^{\infty} q' d\psi\right)^2 - 2\int_{\omega}^{\bar{\omega}} q'\, d\psi \cdot \int_{\bar{\omega}}^{\infty} q'\, d\psi.$$

We shall show below that the function $F(p, \omega, \bar{\omega})$, being sufficiently small for an appropriate $\bar{\omega}$, can be estimated *a priori* via computer simulation.

Denoting $p = q'$, we obtain the boundary value problem

$$D(p) \equiv p' - 2\omega p \int_{\omega}^{\bar{\omega}} p\, d\psi + \left(\int_{\omega}^{\bar{\omega}} p\, d\psi \right)^2 - F(p, \omega, \bar{\omega}) = 0, \quad 0 < z < L, \tag{25}$$

$$p(0; \omega) = \varphi_2(\omega), \tag{26}$$

$$p(L; \omega) = \varphi_3(\omega). \tag{27}$$

Obviously, functions p'' and q''' have the same set of points of discontinuities. However, problem (25)–(27) is still overdetermined. Therefore, it is meaningful to search for its least-squares solution defined as a minimiser of the functional

$$J_0(p) = \int_{\underline{\omega}}^{\bar{\omega}} \int_0^L |D(p)|^2\, dz\, d\omega \tag{28}$$

subject to (26) and (27).

From a theoretical standpoint, the result established above allows the reformulation of the problem of locating the interfaces in terms of minimising (28) subject to (26) and (27). However, there is no guarantee that the functional $J_0(p)$ is strictly convex. In this case, there may exist many local minima, and a search for the global minimum is very difficult to perform. Therefore, we shall reduce the overdetermined problem (25)–(27) to a more simple problem resulting from the physically meaningful estimates of the integral terms $\int_{\omega}^{\bar{\omega}} p\, d\psi$, $\int_{\bar{\omega}}^{\infty} p\, d\psi$.

2.3. *Reduction to the Cauchy Problem for a Riccati Equation*

Since the function $p''(z; \omega)$ is discontinuous at every point z_ι for all $\omega > 0$, we fix a certain $\omega = \omega_0$. According to the mean-value theorem for integrals, there exists a certain $\hat{\omega} \in [\omega_0, \bar{\omega}]$ such as

$$p(z; \hat{\omega}) = \frac{1}{\bar{\omega} - \omega_0} \int_{\omega_0}^{\bar{\omega}} p(z; \psi)\, d\psi. \tag{29}$$

On the other hand, the function $p(z; \hat{\omega})$ can obviously be represented in the form

$$p(z; \hat{\omega}) = p(z; \omega_0) + \Delta p(z; \omega_0, \hat{\omega}), \quad z \in [0, L]. \tag{30}$$

For instance, in computational experiments with low-frequency electromagnetic sounding of layered marine configurations, the C-norm of the function Δp is small compared to the function $p(z; \omega)$ for all $\omega \in [\omega_0, \bar{\omega}]$ (see Fig. 2), and the interval $[\omega_0, \bar{\omega}]$ is small as well. It allows us to estimate the function Δp from computer simulation with the simple background model indicated in Section 3.3 and to use such an estimate when solving the inverse problem for any layered configuration.

We obtain from (29) and (30)

$$\int_{\omega_0}^{\bar{\omega}} p(z;\psi)\,d\psi = (\bar{\omega}-\omega_0)(p(z;\omega_0)+\Delta p). \tag{31}$$

Substituting the representation (31) in (25), we obtain the Riccati equation

$$p' = A(\bar{\omega},\omega_0)p^2 + B(\bar{\omega},\omega_0,\varepsilon,\Delta p)p + C(\bar{\omega},\omega_0,\varepsilon,\Delta p),$$

where

$$A(\bar{\omega},\omega_0) = (\bar{\omega}-\omega_0)(3\omega_0-\bar{\omega}), \tag{32}$$

$$B(\bar{\omega},\omega_0,\varepsilon,\Delta p) = 2(\varepsilon(z)+(\bar{\omega}-\omega_0)\Delta p)(2\bar{\omega}-\omega_0), \tag{33}$$

$$C(\bar{\omega},\omega_0,\varepsilon,\Delta p) = -(\varepsilon(z)+(\bar{\omega}-\omega_0)\Delta p)^2, \tag{34}$$

and $\varepsilon(z) = \int_{\bar{\omega}}^{\infty} p\,d\psi$.

Noticing that the boundary condition (27) does not depend on the frequency sounding data, we make a conjecture that the neglect of this condition does not affect significantly the points of discontinuities of $p''(z;\omega)$, though it certainly affects the least-squares solution. Therefore, we shall consider an auxiliary Cauchy problem,

$$s' = A(\bar{\omega},\omega_0)s^2 + B(\bar{\omega},\omega_0,\varepsilon,\Delta s)s + C(\bar{\omega},\omega_0,\varepsilon,\Delta s), \tag{35}$$

$$s(0;\omega) = \varphi_2(\omega), \tag{36}$$

rather than the overdetermined problem (25)–(27), assuming that the points of discontinuities of $s''(z;\omega)$ are close enough to the points of discontinuities of $p''(z;\omega)$ on the interval $[0, L]$. Thus, the conjecture made above allows us to reformulate the problem of locating the interfaces in a layered medium as follows.

Given the function $u(0,\omega)$, the function $s(z,\omega)$ satisfies the Cauchy problem (35)–(36). *Determine approximately the set Z.*

Unlike the least-squares solution, the solution of the Cauchy problem (35)–(36) can be efficiently computed. Moreover, due to the smallness of the interval $[\omega_0, \bar{\omega}]$, the coefficient $A(\bar{\omega},\omega_0)$ in the Riccati equation (35) is sufficiently small. This means that although the numerical solution of the Cauchy problem for the Riccati equation is, in general, unstable with respect to both the small perturbations of $\varphi_2(\omega)$ and round-off errors, in our case, the manifestations of such an instability are expected to be sufficiently small.

3. COMPUTATIONAL ALGORITHM

Without loss of generality, we assume that the functions $u(0;\omega)$, $s(z;\omega_0)$ are approximated by the corresponding discrete functions $u_j = u(0;\omega_j)$, $s_k = s(z_k;\omega_0)$ on the uniform grids

$$G_z = \{z_k\colon z_k = h_z(k-1),\ h_z = L/(N-1),\ k = 1,2,\ldots,N\},$$

$$G_\omega = \{\omega_j\colon \omega_j = \underline{\omega} + h_\omega(j-1),\ h_\omega = (\bar{\omega}-\underline{\omega})/(M-1),\ j = 1,2,\ldots,M\}.$$

Given u_J, ε_J, and the estimate of Δs_J, $\bar{\omega}$, ω_0, estimate $\{z_\iota\}_{\iota=1}^n$.

Step 1. Compute the Cauchy condition φ_{2J} using (17) at $\omega = \omega_0$. Since the function $u(0;\omega)$ is, in general, given approximately, we employ the optimal regularisator [18] for the numerical differentiation (see the brief description below).

Step 2. Compute the coefficients A, B, C using (32)–(34).

Step 3. Compute the particular solution $\tilde{s}$ of the Cauchy problem (35)–(36) using the Adams–Bashforth–Moulton scheme of the predictor–corrector method of the fourth order. In accordance with this scheme, the predictor is constructed as

$$\tilde{s}_{k+1} = \tilde{s}_k + \frac{h_z}{12}(23\Phi_k - 16\Phi_{k-1} + 5\Phi_{k-2}) + O\left(h_z^4\right), \quad (k = 3, 4, \ldots, N-1), \tag{37}$$

and the corrector has the form

$$\tilde{s}_{k+1} = \tilde{s}_k + \frac{h_z}{12}(5\Phi_{k+1} + 8\Phi_k - \Phi_{k-1}) + O\left(h_z^4\right), \quad (k = 2, 3, \ldots, N-1), \tag{38}$$

where $\tilde{s}_k = \tilde{s}(z_k)$ and $\Phi_k = A\tilde{s}_k^2 + B\tilde{s}_k + C$. It is clear that $\tilde{s}_1 = \varphi(\omega_0)$. The values $\tilde{s}_k$ $(k = 2, 3)$ in (37) are computed as

$$\tilde{s}_k = \tilde{s}_{k-1} + \int_{z_{k-1}}^{z_k} \Phi(\xi, \tilde{s})\, d\xi.$$

In (38), the value $\tilde{s}_2$ is taken directly from (37), and the others are the admissible roots of the quadratic equation with respect to $\tilde{s}_{k+1}$.

Step 4. Compute the second derivative $\tilde{s}''$ of the particular solution $\tilde{s}$ by the optimal operator for the numerical differentiation.

Step 5. Determine approximately the points of discontinuities $\{z_\iota\}_{\iota=1}^n$, analising the critical points of the discrete function $\tilde{s}''$. Specifically, we adopt the following procedure. Recall that the points of discontinuities of q''' are the so-called corner points. This means that the inequality $\mathbf{t}_+(z_\iota) \neq \mathbf{t}_-(z_\iota)$ is valid for all z_ι. Here, $\mathbf{t}_+(z_\iota) = \lim_{z\to z_\iota+} \mathbf{t}$, $\mathbf{t}_-(z_\iota) = \lim_{z\to z_\iota-} \mathbf{t}$, and $\mathbf{t}$ is the unit tangential vector. However, due to the regularising property of the optimal operator, these points are indicated as the smoothed corner points. Let $\tilde{z}_\iota$ be any such point, and let $[\tilde{z}_\iota - r, \tilde{z}_\iota + r]$, $r > 0$, be a sufficiently small area of this point. Choosing any two couple of points $(z, \tilde{s}(z))$ from the left and from the right of every critical point, we derive the equations for a couple of straight lines that are collinear to the corresponding tangential vectors. Finally, the point of intersection of these lines gives us the estimate of z_ι. Although in our computational experiments the smoothed corner points have been interactively detected, this procedure can be authomised.

Now, we briefly describe the optimal regularisator R_{opt} approximating the operator $D \equiv \partial/\partial z$. Note that in accordance with [18], the optimal regularisator is defined as a minimiser of the residual $\|\partial/\partial z - R\|$ on a certain class of all linear operators R acting in the L_2 space. Let $f_\delta(z)$ be a certain approximation of $f(z)$, such as $\|f_\delta - f\|_{C[0,L]} \leq \delta$, where $\delta > 0$ is the known positive number. Also, we assume that the function $f(z)$ belongs to the set

$$M = \{f(z): f'' \in C[0, L], \|f''\| \le m, m > 0\}.$$

Then, it follows directly from [18] that the optimal regularisator is given by

$$R_{opt} f_\delta = \frac{f_\delta(z + H(\delta, m)) - f_\delta(z - H(\delta, m))}{2H(\delta, m)}, \tag{39}$$

where $H(\delta, m) = \sqrt{\frac{2\delta}{m}}$ is the "regularising" step. In computational practice, the parameter δ and m are determined from computer simulatin.

4. NUMERICAL EXAMPLES

In this section we perform some computational experiments with simulated data using both the forward and the inverse models of electromagnetic frequency sounding of layered media.

4.1. *Model Problems*

As a forward model problem, we consider the boundary value problem (8)–(10). For the purpose of computing, we rewrite this problem in dimensionless variables $\alpha = \omega/\omega_{min}$, $\xi = z/2L$ as

$$u''(\xi; \alpha) + \hat{k}^2(\xi; \alpha)u(\xi; \alpha) = 0, \quad 0 < \xi < 1/2, \tag{40}$$

$$u'(0; \alpha) = 1, \tag{41}$$

$$u'(1/2; \alpha) - \imath \hat{k}_b u(1/2; \alpha) = 0, \tag{42}$$

where $k^2(\xi, \alpha) = 4L^2 \imath \mu \omega_{min} \alpha \sigma(\xi)$, $k_b = \sqrt{2} L(1 + \imath)\sqrt{\omega_{min} \alpha \mu \sigma_b}$. In computations, we introduce the function $t(\xi; \alpha) = -u'(\xi; \alpha)/u(\xi; \alpha)$, reducing this problem to the Cauchy problem for the Riccati equation. As an isotropic conductive medium, we consider a typical marine configuration consisting of the highly conductive seawater layer and conductive near-seafloor sediments, so that the inhomogeneous layer $0 < \xi < 1/2$ consists of several layers. Then, the conductivity profile $\sigma(\xi)$ is given by

$$\sigma(\xi) = \begin{cases} 0, & \text{for } \xi < 0, \\ \sigma_1, & \text{for } 0 < \xi < \xi_1, \\ \cdots & \cdots \\ \sigma_\imath, & \text{for } \xi_{\imath-1} < \xi_\imath < \xi_\imath, \\ \cdots & \cdots \\ \sigma_n, & \text{for } \xi_n < \xi < 1/2, \\ \sigma_b, & \text{for } \xi > 1/2. \end{cases} \tag{43}$$

We formulate an inverse model problem as estimating the positions $\{z_\imath\}_{\imath=1}^n$ via the solution of the Cauchy problem (35)–(36), in which we replace formally the variables ω, ω_0, z with the variables α, α_0, ξ, respectively. The Cauchy condition becomes $\varphi(\alpha) = -4L\partial(u^{-1}(0; \alpha))/\partial\alpha$. In computational experiments, the derivative over α has been computed via the optimal regularisator (39) assuming that the level δ of algorithmic errors in computing the frequency sounding data $u(0;\alpha)$ does not exceed 5×10^{-4}. In this connection, we did not model any random perturbations (noise) of these data.

4.2. *Scheme of Computational Experiments*

Three specific marine configurations of the same inhomogeneous layer where $L = 75$ m have been used in computational experiments. Such configurations are typical for the Baltic sea. Both interface positions at $z = 0$ m and at $z = L$ m were assumed to be known, and all configurations contain air ($z < 0$ m), seawater, sediment layers, and bedrock ($z > 75$ m). The first configuration contains only one sediment layer 40 m $< z <$ 75 m, and the corresponding conductivity profile is given by

$$\sigma_I(z) = \begin{cases} 0 \text{ S/m}, & \text{for } z < 0 \text{ m}, \\ 0.8 \text{ S/m}, & \text{for } 0 \text{ m} < z < 40 \text{ m}, \\ 0.2 \text{ S/m}, & \text{for } 40 \text{ m} < z < 75 \text{ m}, \\ 0.001 \text{ S/m}, & \text{for } z > 75 \text{ m}. \end{cases}$$

The second configuration is characterised by the conductivity profile

$$\sigma_{II}(z) = \begin{cases} 0 \text{ S/m}, & \text{for } z < 0 \text{ m}, \\ 0.8 \text{ S/m}, & \text{for } 0 \text{ m} < z < 40 \text{ m}, \\ 0.2 \text{ S/m}, & \text{for } 40 \text{ m} < z < 55 \text{ m}, \\ 0.08 \text{ S/m}, & \text{for } 55 \text{ m} < z < 75 \text{ m}, \\ 0.001 \text{ S/m}, & \text{for } z > 75 \text{ m}; \end{cases}$$

i.e., it contains two sediment layers. The third configuration includes four sediment layers, and the conductivity profile is given by

$$\sigma_{III}(z) = \begin{cases} 0 \text{ S/m}, & \text{for } z < 0 \text{ m}, \\ 0.8 \text{ S/m}, & \text{for } 0 \text{ m} < z < 40 \text{ m}, \\ 0.2 \text{ S/m}, & \text{for } 40 \text{ m} < z < 55 \text{ m}, \\ 0.08 \text{ S/m}, & \text{for } 55 \text{ m} < z < 60 \text{ m}, \\ 0.4 \text{ S/m}, & \text{for } 60 \text{ m} < z < 65 \text{ m}, \\ 0.08 \text{ S/m}, & \text{for } 65 \text{ m} < z < 75 \text{ m}, \\ 0.001 \text{ S/m}, & \text{for } z > 75 \text{ m}. \end{cases}$$

It can be noticed that the third configuration contains the thin highly conductive sediment layer 60 m $< z <$ 65 m situated between two relatively poorly conductive layers. From a geological standpoint, such a configuration may be due to the porosity stratification. In other words, in the third configuration, the sediment layer 55 m $< z <$ 75 m with the conductivity of 0.08 S/m may also be stratified over porosity. To reflect this, we have introduced this configuration. To our knowledge, the detection of a thin highly conductive layer by any existing method of electromagnetic sounding is a difficult problem.

Given the conductivity profile (43), we first simulated the frequency sounding data $u(0;\alpha)$ using the Riccati solver for the forward Cauchy problem. Due to the high conductivity of seawater, the frequency $\underline{\omega} = 1\ rad \cdot Hz$ and $\bar{\omega} = 200\ rad \cdot Hz$ have been chosen. After the data were computed, we made a reduction to the Cauchy problem (35)–(36) as it was described in Section 1.3, estimated the quantities $\varepsilon(\xi)$, $\Delta s(\xi)$, α_0, and applied the algorithm described in Section 3 for locating the interfaces.

4.3. *Numerical Results*

To estimate the function $\varepsilon(\xi)$, we first computed the solution $u(\xi;\alpha)$ of the forward problem (40)–(42) for all three configurations indicated above using the broader frequency range $[\underline{\omega}, \Omega_0]$ *rad · Hz*, where $\Omega_0 = 5000$ *rad · Hz*. Next, since the function $q'(\xi;\alpha)$ can be expressed in the form

$$q' = \alpha^{-1}\left[u^{-1}\frac{\partial^2 u}{\partial\alpha\partial\xi} - u^{-2}\frac{\partial u}{\partial\xi}\cdot\frac{\partial u}{\partial\alpha}\right] - \alpha^{-2}u^{-1}\frac{\partial u}{\partial\xi},$$

we precomputed the function $\varepsilon(\xi) = \int_{\bar{\alpha}}^{\Omega_0} q'\,d\psi$ via quadratures. Figure 1 shows the results of computations. The estimates of the remainder term for the second and third configurations are almost the same.

Analysing these results, we notice that an appropriate estimate of $\varepsilon(\xi)$ can be obtained from computer simulation of the three-layer background configuration containing the air,

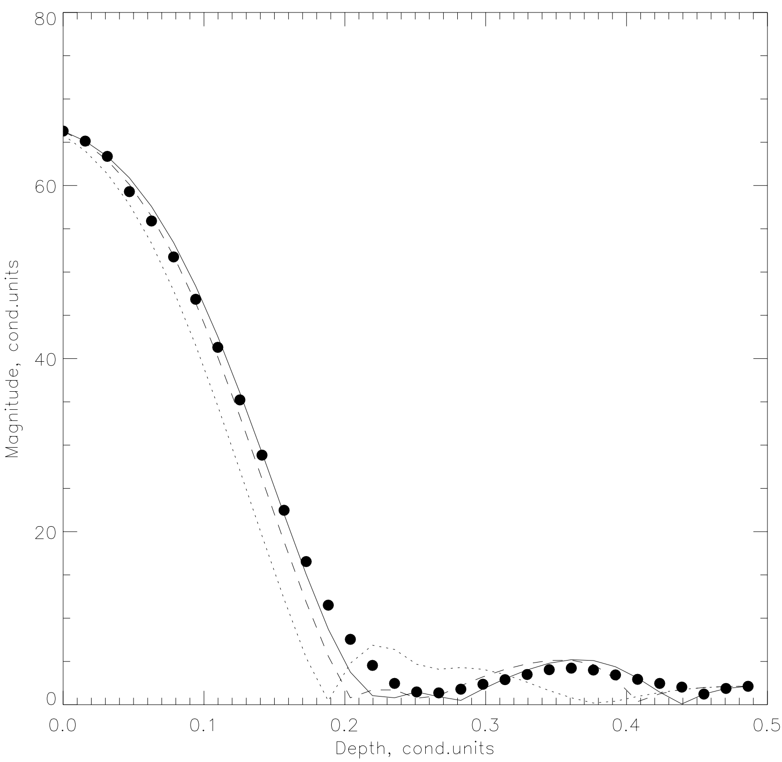

FIG. 1. The remainder integral $\varepsilon(\xi)$ for the first (solid line), second (dotted line), and third (dashed line) configurations. The results for the three-layer configuration are shown by the filled bullets.

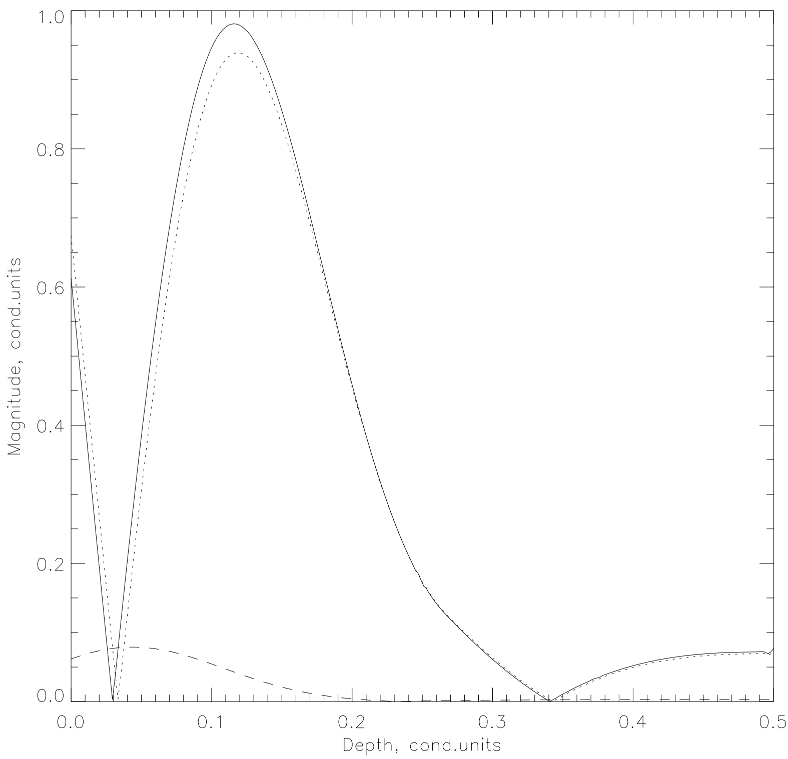

FIG. 2. The results of computer simulation of the function Δp for the background configuration. The plot of $|p(\xi; \alpha_0|$ is shown by the solid line. The dotted line corresponds to the plot of $|p(\xi; \hat{\alpha}|$, and the dashed line corresponds to the plot of $|\Delta p|$.

seawater, and bedrock, and the background conductivity profile is given by

$$\sigma_0(z) = \begin{cases} 0 \text{ S/m}, & \text{for } z < 0 \text{ m}, \\ 0.8 \text{ S/m}, & \text{for } 0 \text{ m} < z < 40 \text{ m}, \\ 0.001 \text{ S/m}, & \text{for } z > 40 \text{ m}. \end{cases}$$

Such *a priori* information is often available in practice, and the forward problem (40)–(42) can be analytically solvable for $\sigma_0(z)$ prior to locating the interfaces.

To choose an appropriate α_0 and estimate the function Δs, we computed the function $p(\xi; \alpha)$ for the background and three other configurations using the dimensionless analogues of Eqs. (29) and (30). Figure 2 shows the results of such computations for the background configuration and $\alpha = 0.95$. In this computational experiment, we employed the frequency range $[1, 200]$ $rad \cdot Hz$. The results for the other three configurations are almost identical, and therefore, they are not shown in Fig. 2. Thus, the results of computational experiments show that the appropriate estimates of the functions $\varepsilon(\xi)$, Δs can be obtained from the forward modelling of the background three-layer configuration prior to locating the interfaces. In the next computational experiments, we used the function $|\Delta p|$ indicated in Fig. 2 as an estimate of Δs.

We applied the algorithm indicated in Section 3 to the simulated frequency sounding data. Figures 3, 4, and 5 show the regularised second derivatives of the function $s(\xi; \alpha_0)$ computed for the first, second, and third configurations, respectively. It is clearly shown

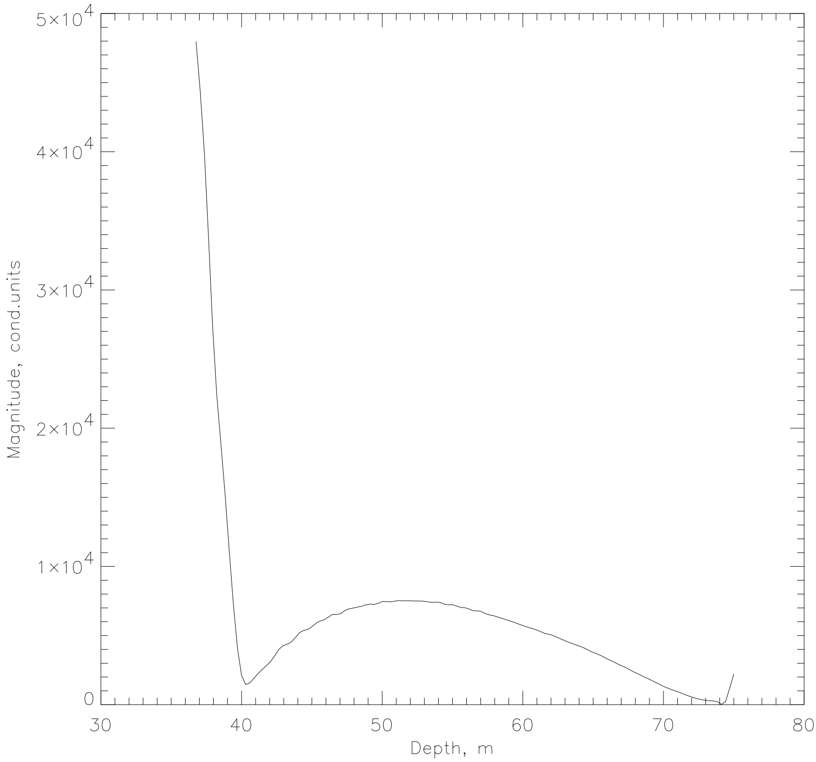

FIG. 3. The regularised second derivative of $s(\xi; \alpha_0)$ for the first configuration.

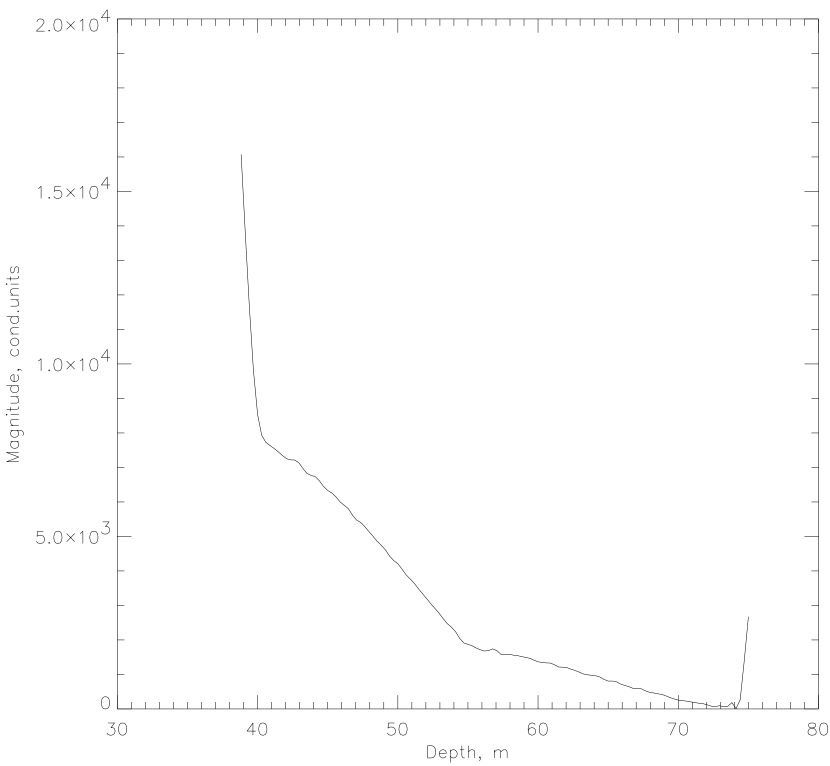

FIG. 4. The regularised second derivative of $s(\xi; \alpha_0)$ for the second configuration.

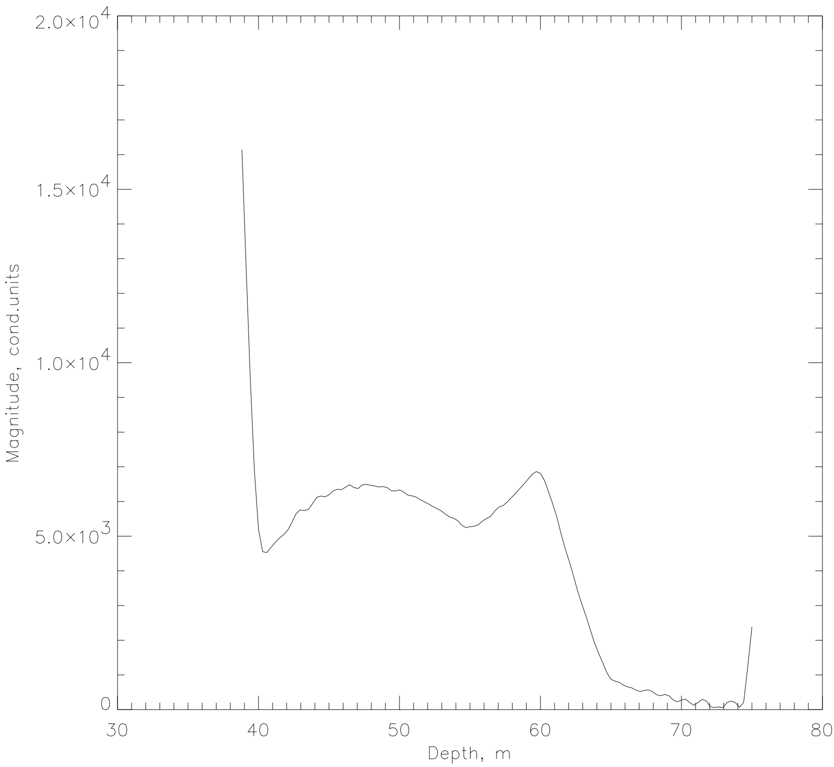

FIG. 5. The regularised second derivative of $s(\xi;\alpha_0)$ for the third configuration.

that all these derivatives have the smoothed corner points, and the number of such points equals the number of interfaces between the sediment layers. The positions of interfaces determined in accordance with *Step 5* of the algorithm are as follows: (i) $\tilde{z}_1 = 39.71$ m for the first configuration, (ii) $\tilde{z}_1 = 39.71$ m, $\tilde{z}_2 = 55.00$ m, and (iii) $\tilde{z}_1 = 39.71$ m, $\tilde{z}_2 = 55.00$ m, $\tilde{z}_3 = 59.71$ m, $\tilde{z}_4 = 64.41$ m. In all cases, the relative errors do not exceed 1%. It should also be pointed out that in spite of use of the optimal operator for computing both the first and the second derivatives of $s(\xi;\alpha_0)$, the latter one, being actually a function of jumps, may contain some large oscillations situated in the seawater layer because of the largest gradient of $s'(\xi;\alpha_0)$, due to the skin effect. These oscillations could also be interpreted as spurious layers. Fortunately, in all marine configurations, the depth of the water column can be roughly estimated from the hydrographic maps, so we can exclude the part of $s''(\xi;\alpha_0)$ associated with the water column and analyse the second derivative starting with a sufficiently small area of the interface between the seawater and the first sediment layer. For the same reason, we ignored the slight oscillations and a jump at the small area of the interface between the sediments and bedrock. For the other models, however, the analysis of the second derivative of $s(\xi;\alpha_0)$ requires special care.

5. CONCLUSIONS

We have presented an efficient algorithm for the problem of locating the interfaces arising in frequency sounding of layered media. This algorithm has resulted from our main finding that the points of discontinuity of the third derivative of the solution of a certain auxiliary boundary value problem are associated with the interface positions. The transition from the original boundary value problem to such a problem was made using the identical transformations introduced previously by one of authors in connection with applying Carleman's estimates to the coefficient inverse problems.

Unlike the conventional nonlinear least-squares formulation, the proposed algorithm does not require the time-consuming nonlinear optimisation procedure and it can be effectively implemented via the Riccati solver. We have illustrated its feasibility in several computational experiments with electromagentic frequency sounding of layered marine configurations. This algorithm has proven computationally to be very efficient at locating the interfaces between near-seafloor sediment layers. We believe that a similar algorithm can be constructed for acoustic frequency sounding of oceanic waveguides. We also believe that the proposed approach can be successfully applied to optical sensing at near-infrared frequencies. We reserve these issues for further publications.

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